18.312: Algebraic Combinatorics — Lionel Levine

Lecture 17

Lecture date: April 14, 2011 — *Notes by: Santiago Cuellar*

Todays topics:

1. König's Theorem
2. Kasteleyn's Theorem: Domino tilings for planar regions (matching planar graphs)

1 Back to König's Theorem

Theorem 1 (Reformulation of Hall's marriage theorem) *Given sets $I_1, I_2, \dots I_n \subseteq [n]$ suppose that:*

$$|I_{i_1} \cup I_{i_2} \cup \cdots \cup I_{i_k}| \geq k \qquad \text{for all } 1 \leq i_1 < i_2 < \cdots < i_k \leq n. \tag{1}$$

Then $\{I_1, I_2, \dots I_n\}$, has a transversal *(or a* system of distinct representatives*). That is, there exists a permutation $\sigma \in S_n$ such that $\sigma(i) \in I_i$ for all $i = 1, \dots, n$.*

Proof: Construct bipartite graph on vertex set $V = X \cup Y$ where $X = \{I_1, I_2, \dots I_n\}$, $Y = [n]$ and the edges $E = \{(I_i, j) |\ j \in I_i\}$. Then (1) becomes:

$$\Gamma(\{I_{i_1}, I_{i_2}, \dots, I_{i_k}\}) = \#(I_{i_1} \cup I_{i_2} \cup \cdots \cup I_{i_k}) \geq k$$

Then, by Hall's marriage theorem, there is a matching which implies a transversal. □

Slight generalization 1 $I_1, I_2, \dots I_n \subseteq [m]$, *If (1) holds (note that this implies $n \leq m$) then there is an injective map $\sigma : [n] \to [m]$ such that $\sigma(i) \in I_i$ for all $i = 1, \dots, n$.*

Recall the König's theorem restated as a theorem over bipartite graphs:

Theorem 2 (König) *Given a bipartite graph $G = (V, E)$:*

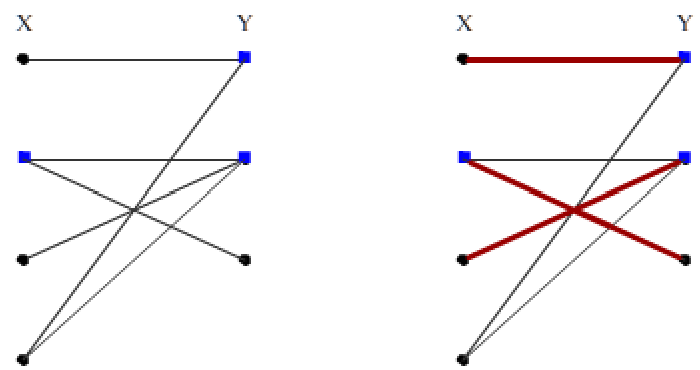
$$\min_{\text{vertex cover } C} |C| = \max_{\text{matchings } M} |M| \tag{2}$$

Proof:[continued from last lecture]

It is easy to show $|C| \geq |M|$, because each edge of the matching, covers at least one vertex from the cover.

Now we try to prove $\min |C| \leq |M|$ for some matching M. Given a minimal vertex cover C, we want to extend into a matching M_C. Suppose the X and Y are the two components of the graph then let $C_X = C \cap X$ and $C_Y = C \cap Y$.

Example 3



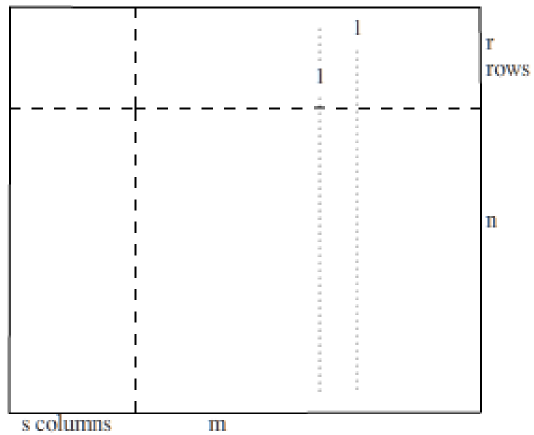
Consider the induced subgraph $G' = (V', E')$ with vertex set $V' = C_X \cup (Y - C_Y)$.

Claim 4 *Given a subset $A \subseteq C_X$. Then $\#\Gamma'(A) \geq \#A$*

Proof: For the sake of contradiction, assume $\#\Gamma'(A) < \#A$, then $(C - A) \cup \Gamma'(A)$ would be a smaller vertex cover. □

Likewise, $G'' = (V'', E'')$, with vertex set $V'' = C_Y \cup (X - C_X)$, has a matching M'' using all the vertices of C_X. Then $M = M' \cup M''$ is the matching we were looking. Since $M' \cap M'' = \emptyset$, then $|M| = |M'| + |M''| = |C_X| + |C_Y| = C$ □

Example 5 *Consider a $m \times n$ rectangular matrix with entries 1 and 0. We look for a subset of rows and columns that covers all the 1's.*



Let $r + s$ is the min number of lines of the matrix containing all the 1's of M. Without loss of generality we can assume all the chosen columns are to the left and the rows are at the top. Consider each chosen row, it has to have a 1 on the right (i.e. not on the chosen columns). Moreover, for two rows they have a 1 in different columns, otherwise they could be replaced by a column covering both. Likewise, the columns satisfy a similar property which gives us a set of $r + S$ 1's.

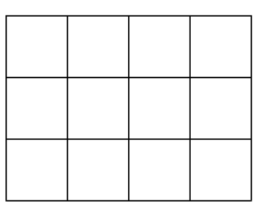
2 Kasteleyn's Theorem: Domino tilings for planar regions

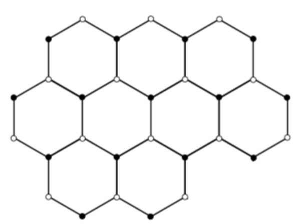
Definition 6 *A graph $G = (V, E)$ is called* planar *if there exists a function $\alpha : V \to \mathbb{R}^2$ and for all $(i, j) \in E$ there is another continuous, injective function $\gamma_{i,j} : [0, 1] \to \mathbb{R}^2$ such that $\gamma_{i,j}(0) = \alpha(i)$, $\gamma_{i,j}(1) = \alpha(j)$ and $\gamma_{i,j}(0, 1) \cap \gamma_{i',j'}(0, 1) = \emptyset$ for different edges.*

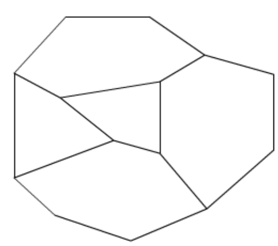
Intuitively, a graph is planar if you can draw it on the plane with no intersection of edges.

Bipartite planar graphs:

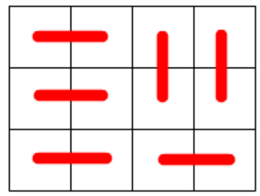
1. Square Grid
2. Hexagonal Lattice
3. General



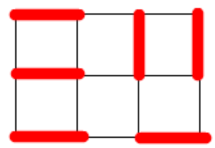




Question 7 *How many ways can you tile an $m \times n$ square grid by 2×1 dominoes?*

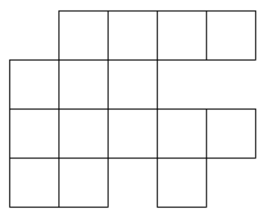


We consider the problem on the dual graph:



So the question is equivalent to find the number of perfect matchings in the dual graph.

Let G be a finite induced subgraph of $\mathbb{Z}^2$.



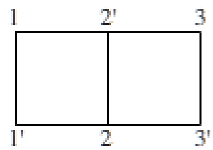
Definition 8 *The* Kasteleyn matrix *of G is the $V \times V$ matrix:*

$$K_{u,v} = \begin{cases} 1 & u, v \text{ is an horizontal edge} \\ i = \sqrt{-1} & u, v \text{ is a vertical edge} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 9 (Kasteleyn) *Given the graph defined above and it's Kasteleyn matrix:*

$$\#\{\text{perfect matchings of } G\} = \sqrt{|\det K|}$$

Example 10 $(m = 2, n = 3)$



$$K = \begin{bmatrix} 0 & 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & 1 & i & 1 \\ 0 & 0 & 0 & 0 & 1 & i \\ i & 1 & 0 & 0 & 0 & 0 \\ 1 & i & 1 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 \end{bmatrix}$$

Since we only care about the absolute value of the determinant we can swap columns to get:

$$K = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right] \text{ where } A = \begin{bmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 1 & i \end{bmatrix}$$

Then $|\det K| = |\det^2 A| = |i^3 - i - i|^2 = 9$. *So there are 3 matchings which can be checked by looking at the match for vertex* 2, *it has three possibilities and the rest of the matches are uniquely defined afterwards.*

Proof:[Kasteleyn's theorem] G is a bipartite graph with parts X and Y. Let

$$w(u,v) = \begin{cases} 1 & \text{if } (u,v) \text{ is a horizontal edge} \\ i & \text{if } (u,v) \text{ is a vertical edge} \\ 0 & \text{otherwise.} \end{cases}$$

So

$$K = \left[\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right] \text{ where } A_{u,v} = \begin{cases} w(u,v) & u,v \in E \\ 0 & \text{otherwise} \end{cases}$$

By swapping the columns like in the example we get, $|\det K| = |\det^2 A| = |\det A|^2$. From this, it is enough to show that $|\det A| = \#\{\text{perfect matchings of } G\}$.

We have $n = |X| = |Y|$ and $X = \{u_1, u_2, \ldots u_n\}$, $Y = \{v_1, v_2, \ldots v_n\}$

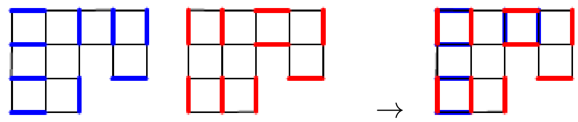
$$\det(A) = \sum_{\sigma \in S_n} (-1)^\sigma w(1, \sigma(1)) \cdot \ldots w(n, \sigma(n)) \quad (3)$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma w(u_1, v_{\sigma(1)}) \cdot \ldots w(u_n, v_{\sigma(n)}) \quad (4)$$

Notice that in the last part, the summands are 0 if and only if one of the w are 0. That is they are not 0 when σ represents a matching, so it's only left to check that some of the matchings don't cancel. In other words

Claim 11 *Any two matchings occur with the same sign in the sum (There is no cancelation)*

To do this, we consider two distinct matchings M and M'



Which together can be viewed as cycles ($M \cup M'$ is a disjoint union of even cycles.)

Lemma 12 *Let $v_1, v_2 \dots v_{2k}$ be a cycel in $\mathbb{Z}^2$. Let*

$$\pi = \Big(\prod_{i\ odd} w(v_i, v_{i+1})\Big) / \Big(\prod_{i\ even} w(v_i, v_{i+1})\Big)$$

then $\pi = (-1)^{k+l-1}$, where l is the number of points in $\mathbb{Z}^2$ strictly enclosed in the cycle.

Proof:

We prove the lemma by induction over the area enclosed by the cycle.

- The base case is when no area is enclosed.

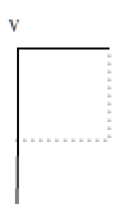
  

  Notice that only this case is possible for area no area enclosed because the cycle is the union of two matchings, so every vertex has degree exactly 2. In this case

  $$\frac{w(v_1, v_2)}{w(v_2, v_1)} = 1 = (-1)^{1+0-1}$$

- Inductive step: Without loss of generality we can assume v_1 is the topmost vertex in the leftmost column. According to this we consider three cases

  1. So the new variables are:



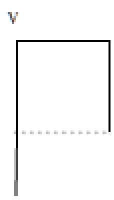
$k' = k$
$l' = l - 1$
$V' = V + 1$
$H' = H - 1$

$V = \#\text{vertical edges}$
$H = \#\text{horizontal edges}$

Hence we get, using our inductive hypothesis,

$$\begin{aligned} \pi &= \pi' \frac{-i}{i} && (5) \\ &= -(-1)^{k'+l'-1} && (6) \\ &= (-1)^{k+l-1} && (7) \end{aligned}$$

2. The new variables are:



$k' = k - 1$

$l' = l$

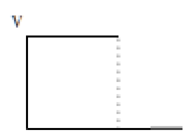
$V' = V$

$H' = H - 2$

Hence we get, using our inductive hypothesis,

$$\begin{aligned} \pi &= \pi' \frac{1}{i^2} && (8) \\ &= -(-1)^{k'+l'-1} && (9) \\ &= (-1)^{k+l-1} && (10) \end{aligned}$$

3. This last case is analogous to the second case:



This proves the lemma.□

We will use the lemma, to prove the previous claim and finish the proof of the theorem in the next lecture. □