18.312: Algebraic Combinatorics	Lionel Levine
Lecture 18	
Lecture date: Apr 14, 2011	*Notes by: Taoran Chen*

1 Kasteleyn's theorem

Theorem 1 (Kasteleyn) *Let G be a finite induced subgraph of $\mathbb{Z}^2$. Define the Kasteleyn matrix of G to be the $V \times V$ matrix:*

$$K_{u,v} = \begin{cases} 1 & (u,v) \text{ is a horizontal edge} \\ i & (u,v) \text{ is a vertical edge} \\ 0 & \text{else} \end{cases}$$

then

$$\#\{\text{perfect matchings of } G\} = \sqrt{|\det K|}$$

Proof:[continued] It suffices to show that any two nonzero terms in the expression

$$\det A = \sum_{\sigma \in S_n} (-1)^\sigma w(u_1, v_{\sigma(1)}) w(u_2, v_{\sigma(2)}) ... w(u_n, v_{\sigma(n)})$$

have the same sign. Given two perfect matchings M,M' of G, they correspond to some permutations (say,σ and σ' respectively) and some nonzero terms in the expression above. Their union $M \cup M'$ is a disjoint union of even cycles, so we can transform M into M' by rotating the edges along each cycle in turn. It suffices to show that rotation along a single cycle does not affect the sign of the corresponding summand. In particular, we only need to consider the case when $M \cup M'$ is a single cycle.

Let $M \cup M'$ be the cycle $u_1, v_1, u_2, v_2 ..... u_n, v_n$, where $(u_1, v_1), (u_2, v_2) ... (u_n, v_n)$ being edges of M and $(u_1, v_n), (u_2, v_1) ... (u_n, v_{n-1})$ being edges of M'. Then σ is the identity permutation, and $\sigma' = (n, n-1, ..., 1)$ is the cyclic permutation having length n, thus $(-1)^\sigma = 1$ and $(-1)^{\sigma'} = (-1)^{n-1}$. By a lemma from the last lecture,

$$\begin{aligned} \frac{w(u_1, v_{\sigma(1)}) w(u_2, v_{\sigma(2)}) ... w(u_n, v_{\sigma(n)})}{w(u_1, v_{\sigma'(1)}) w(u_2, v_{\sigma'(2)}) ... w(u_n, v_{\sigma'(n)})} &= \frac{w(u_1, v_1) w(u_2, v_2) ... w(u_n, v_n)}{w(v_1, u_2) w(v_2, u_3) ... w(v_n, u_{n-1})} \\ &= (-1)^{n+l-1} \end{aligned}$$

where l is the number of vertices enclosed by $M \cup M'$. Since the interior of $M \cup M'$ is a disjoint union of even cycles, l is even. As a consequence, ratio of sign for M and sign for M' is $(-1)^{n+l-1}/(-1)^{n-1} = 1$, which completes the proof. □

2 Domino tilings of a $m \times n$ rectangle

As an application of the Kasteleyn's theorem, we compute the number of tilings by 2×1 domino of a $m \times n$ rectangle, which is equivalent to find the number of perfect matchings of the dual graph, G.

Definition 2 *Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$,define $G_1 \times G_2$ to be the graph having the following properties:*

- *The vertex set of $G_1 \times G_2$ is $V_1 \times V_2$*
- *Two vertices (u_1, u_2) and (v_1, v_2) of $G_1 \times G_2$ are connected by an edge if and only if either $(u_1, v_1) \in E_1$ or $(u_2, v_2) \in E_2$*

Definition 3 *Let $G = (V, E)$, the **adjacency matrix**,A, is the $V \times V$ matrix such that*

$$A_{u,v} = \begin{cases} 1 & (u,v) \in E \\ 0 & else \end{cases}$$

We begin our analysis by finding the eigenvalues of the adjacency matrix of the path graph P_n.

pathgraph.png
P_6

Proposition 4 *Let A_n be the adjacency matrix of the path graph P_n. The eigenvalues of A_n are $2\cos\frac{\pi j}{n+1}$ for $j = 1, 2, ..., n$.*

Proof: The adjacency matrix A_n has the form:

$$A_n = \begin{bmatrix} 0 & 1 & 0 & 0\cdots & 0 \\ 1 & 0 & 1 & 0\cdots & 0 \\ 0 & 1 & 0 & 1\cdots & 0 \\ & & \ddots & \vdots & \\ 0 & 0\cdots & 1 & 0 & 1 \\ 0 & 0\cdots & 0 & 1 & 0 \end{bmatrix}$$

We know that λ is an eigenvalue of A_n if and only if there exists a nonzero vector $v = (v_1, v_2, ..., v_n)^t$ such that $A_n v = \lambda v$. Writting the condition $A_n v = \lambda v$ in coordinates, we obtain the system of equations

$$\begin{cases} v_2 & = & \lambda v_1 \\ v_1 + v_3 & = & \lambda v_2 \\ v_2 + v_4 & = & \lambda v_3 \\ \cdots & & \\ v_{n-1} & = & \lambda v_n \end{cases}$$

If we make the convention that $v_0 = 0 = v_{n+1}$, the system of equation becomes the linear recurrence $v_{i+1} + v_{i-1} = \lambda v_i$, $1 \le i \le n$. Since the linear recurrence can also be written as $(E^2 - \lambda E + 1)v = 0$, its solution has the form $v_i = a\alpha^i + b\beta^i$ (unless $\alpha = \beta$), where α,β are the solutions of the equation $x^2 - \lambda x + 1 = 0$. In particular, $\alpha\beta = 1$, $\alpha + \beta = \lambda$. From the initial data $v_0 = 0 = v_{n+1}$, we deduce $\alpha^{n+1} = \beta^{n+1}$. This, along with the equation $\alpha\beta = 1$, gives us

$$\begin{cases} \alpha^{2n+2} & = & 1 \\ \beta & = & \frac{1}{\alpha} \end{cases}$$

hence α is some $(2n+2)^{th}$ root of unity. Consequently,

$$\lambda = \alpha + \beta = 2\mathrm{Re}(\alpha) = 2\cos\frac{\pi j}{n+1}, \qquad j = 0, 1, ..., 2n+1.$$

Since $2\cos\frac{\pi j}{n+1} = 2\cos\frac{\pi(2n+2-j)}{n+1}$, we need only to consider the possibilities $j = 0, 1, 2, ..., n+1$. If $j = 0$, $\lambda = 2$, the equation $x^2 - \lambda x + 1 = 0$ has root $x = 1$ of multiplicity 2. In this case the v_i has the form $ai + b$. Solving the initial data $v_0 = 0 = v_{n+1}$ we find that v_i is constantly 0, which is forbidden. Similarly, we can show that j cannot be $n+1$. Therefore, the remaining possible values of the eigenvalue λ are $2\cos\frac{\pi j}{n+1}$, $j = 1, 2, ..., n$. A $n \times n$ matrix has exactly n eigenvalues, so we conclude that they are indeed the eigenvalues of A_n. $\square$

The dual graph, G, of the $m \times n$ rectangle can be expressed as $G = P_m \times P_n$, where P_m, P_n are the path graphs. It's not hard to check that the Kasteleyn matrix of G, K, can be written as

$$K = A_m \otimes I_n + i(I_m \otimes A_n)$$

where the symbol $\otimes$ denotes tensor product of matrices, and I_n and I_m are the identity matrices. We are to find the eigenvalues of K.

Proposition 5 *Let the eigenvalues of A_m, A_n be $\mu_k, k = 1, 2, ..., m$ and $\lambda_j, j = 1, 2, ..., n$, respectively. Let w_k, v_j be the associated eigenvectors. Then $\mu_k + i\lambda_j, k = 1, 2, ..., m, j = 1, 2, ..., n$ are the eigenvalues of K, with associated eigenvectors $w_k \otimes v_j$.*

Proof: We check,

$$\begin{aligned}
K(w_k \otimes v_j) &= (A_m \otimes I_n + i(I_m \otimes A_n))(w_k \otimes v_j) \\
&= A_m w_k \otimes v_j + i w_k \otimes A_n v_j \\
&= (\mu_k w_k) \otimes v_j + i w_k \otimes (\lambda_j v_j) \\
&= \mu_k (w_k \otimes v_j) + i\lambda_j (w_k \otimes v_j) \\
&= (\mu_k + i\lambda_j)(w_k \otimes v_j)
\end{aligned}$$

□

Finally, by the Kasteleyn's theorem and the two propositions, we are able to compute the number of domino tilings:

$$\begin{aligned}
\#\{\text{domino tilings}\} &= \#\{\text{perfect matchings of G}\} \\
&= \sqrt{|\det K|} \\
&= (\prod_{k=1}^{m} \prod_{j=1}^{n} |\mu_k + i\lambda_j|)^{1/2} \\
&= (\prod_{k=1}^{m} \prod_{j=1}^{n} (4\cos^2 \frac{k\pi}{m+1} + 4\cos^2 \frac{j\pi}{n+1}))^{1/4}
\end{aligned}$$

3 Matrix-Tree theorem

We begin with a few definitions.

Definition 6 *The* ***Complete graph****, K_n, has vertex set $V = [n]$ and $E = \{(i,j), i \neq j\}$.*

Definition 7 *A* ***spanning subgraph*** *of a graph $G = (V, E)$ is a graph of the form $H = (V, A)$ for some $A \subseteq E$.*

Definition 8 *A graph is* ***connected*** *if for every two vertices $u, v \in V$, G contains a path from u to v.*

Definition 9 *A graph is* ***acyclic*** *if there does not exist $v_0, v_1, ..., v_n = v_0$ such that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, ..., n$. A acyclic graph is also called a* ***forrest****.*

Definition 10 *An acyclic connected graph is called a* ***tree****.*

Definition 11 (verification needed) *Given a finite graph G with n vertices, a spanning subgraph T is called a* ***spanning tree*** *of G if any two of the following conditions are met.*

- *T is connected*
- *T is acyclic*
- *T has $n-1$ edges*

Moreover, any two of the conditions imply the third.

Definition 12 *The* ***complexity*** *of G is $\chi(G) := \#\{\text{spanning trees of } G\}$.*

Theorem 13 (Cayley) $\chi(K_n) = n^{n-2}$

Proof: This will be a special case of the matrix-tree theorem. □

Definition 14 *The* ***Laplacian matrix*** *of G is $L := D - A$, where A is the adjacency matrix and D is given by*

$$D := \begin{bmatrix} d_{v_1} & & & \\ & d_{v_2} & & \\ & & \ddots & \\ & & & d_{v_n} \end{bmatrix}$$

$$d_{v_i} := deg(v_i) = \#\{\text{edges incident to vertex } v_i\}$$

Example 15 *For the complete graph K_4,*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

It's easy to verify that the rows and columns of L sum to 0. In particular, L is a singular matrix, so 0 is one of its eigenvalue.

Theorem 16 (version 1) *Let $G = (V, E)$ be a connected graph such that $|V| = n$, then*

$$\chi(G) = \frac{1}{n}\lambda_1\lambda_2 ... \lambda_{n-1}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the nonzero eigenvalues of L.

Proof will be provided in the next lecture.