18.312: Algebraic Combinatorics	Lionel Levine

Lecture 8

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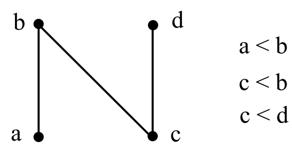
Remark: In the last lecture, someone asked whether all posets could be constructed from a point using the operations of disjoint union, ordinal sum, cartesian product and exponentiation. This is not possible in general. Consider the poset $\mathbf{N} := \{a, b, c, d\}$ with Hasse diagram:



We can see that this poset is not representable as $P + Q$, $P \oplus Q$, $P \times Q$, or P^Q for any posets P and Q.

The class of posets that can be constructed using disjoint union and ordinal sum are called ***series parallel*** posets. In exercise 3.13 of Stanley, it is shown that a poset is series parallel iff its Hasse diagram does not contain $\mathbf{N}$ as a subdiagram.

Lattices

Recall from last lecture the definition of a lattice:

Definition 1 *A poset L is a **lattice** if every pair of elements x, y has*

*(i) a **least upper bound** $x \vee y$ (called **join**), and*

*(ii) a **greatest lower bound** $x \wedge y$ (called **meet**);*

that is

$$z \geq x \vee y \iff z \geq x \text{ and } z \geq y$$
$$z \leq x \wedge y \iff z \leq x \text{ and } z \leq y.$$

For a later result, we will need the following definition.

Definition 2 *An element z of a lattice L is called* ***join irreducible*** *if $z \neq z_1 \vee z_2$ for $z_1, z_2 < z$.*

Observe that the meet and join operations are associative; we note that for meet

$$\begin{aligned} w \leq (x \wedge y) \wedge z \Leftrightarrow w \leq z, x \wedge y \Leftrightarrow w \leq x, y, z \\ \Leftrightarrow w \leq x, y \wedge z \Leftrightarrow w \leq x \wedge (y \wedge z) \end{aligned}$$

and similarly for join

$$\begin{aligned} w \geq (x \vee y) \vee z \Leftrightarrow w \geq z, x \vee y \Leftrightarrow w \geq x, y, z \\ \Leftrightarrow w \geq x, y \vee z \Leftrightarrow w \geq x \vee (y \vee z). \end{aligned}$$

Moreover observe that if L is finite, then L has a unique ***minimal element*** $\hat{0} := \bigwedge_{x \in L} x$ and unique ***maximal element*** $\hat{1} := \bigvee_{x \in L}$. By definition

$$\hat{0} \vee x = x, \quad \hat{0} \wedge x = \hat{0},$$

and

$$\hat{1} \vee x = \hat{1}, \quad \hat{1} \wedge x = x.$$

We might want to show that a poset is a lattice, but only know about one operation. The following lemma tells us that sometimes this is enough.

Lemma 3 *If P is a finite poset such that*

(i) Every $x, y \in P$ have a greatest lower bound.

(ii) P has a unique maximal element $\hat{1}$.

then P is a lattice.

Proof: Consider $x, y \in P$. Let $S = \{z \in P : z \geq x, y\}$. Note that $S \neq \emptyset$ since $\hat{1} \in S$. Take $x \vee y := \bigwedge_{z \in S} z$. We see that

$$z \geq x, y \quad \forall z \in S \Longrightarrow x \vee y \geq x, y,$$

and

$$w \geq x, y \Longrightarrow w \in S \Longrightarrow w \geq \bigwedge_{z \in S} z = x \vee y.$$

This shows that P admits a join operation. □

Notice that by an analogous argument, the same conclusion would follow from the existence of least upper bounds and a unique minimal element. However the meet operation can oftentimes be more natural than the join operation. We will see this in Example 6.

Examples of Lattices

Example 4 (n) *Recall that **n** (handwritten $\underline{n}$) is the set $[n]$ with the usual order relations. We see that $i \wedge j = \min(i,j)$ and $i \vee j = \max(i,j)$.*

Example 5 (Boolean algebras) *Recall that $B_n = \mathcal{P}([n])$. It is a lattice with meet $S \wedge T = S \cap T$, and join $S \vee T = S \cup T$. This example reinforces our notation.*

Example 6 (Partitions) *Let $\pi_n = \{\text{partitions of } [n]\}$ ordered by refinement. Given partitions $\sigma = (\sigma_1, \ldots, \sigma_k)$ and $\tau = (\tau_1, \ldots, \tau_l)$ define $\sigma \wedge \tau$ as*

$$(\textit{nonempty intersections } \sigma_i \cap \tau_j : 1 \leq i \leq k, 1 \leq j \leq l).$$

Since $[n]$ is the unique maximal element of π_n, Lemma 3 implies that π_n is a lattice. Showing that π_n is a lattice without Lemma 3 would be more difficult because the join of two partitions has a messy formula.

Example 7 (Vector Spaces) *Let V be a vector space, and L be the set of linear subspaces ordered by inclusion. L is lattice with meet $S \wedge T = S \cap T$, and join $S \vee T = S + T = \{v + w : v \in S, w \in T\}$.*

Example 8 (Groups) *Let G be a group, and L be the set of subgroups ordered by inclusion. L is a lattice with meet $H \wedge K = H \cap K$, and join $H \vee K = \langle H, K \rangle$.*

Note that Examples 7 and 8 are two cases of lattice constructions common to any algebraic object.

Example 9 (Order-preserving maps) *Let P be a poset, and L be a lattice. Recall that $L^P = \{\text{order preserving maps } P \to L\}$. L^P is a lattice with join*

$$(f \wedge g) : P \ni x \mapsto f(x) \wedge g(x) \in L,$$

and meet

$$(f \vee g) : P \ni x \mapsto f(x) \vee g(x) \in L.$$

Distributive Lattices

In this lecture, we will focus on lattices with a certain property that makes them similar to Boolean algebras. The structure of these lattices is sufficiently nice to allow for a classification theorem (cf. Theorem 15).

Definition 10 *A lattice L is **distributive** if the meet and join operations distribute over each other i.e. for $x, y, z \in L$*

(i) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(ii) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Notice that a Boolean algebra is distributive since union and intersection distribute over each other.

Definition 11 *Given a poset P, an **order ideal** of P is a subset $I \subset P$ such that for all $x \in I$, if $y \leq x$, then $y \in I$.*

Consider for instance the following order ideal of B_3 expressed in terms of it Hasse diagram:

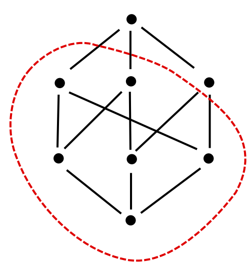


Figure 1: Order ideal of B_3

We will denote the set of order ideals as $J(P)$. Note that $J(P)$ is a poset under inclusion. We will call an order ideal $I \subset P$ ***principal*** if it is of the form $I = \langle x \rangle := \{y \in P : y \leq_P x\}$ for some $x \in P$.

Example 12 (Boolean algebras) *Let $+_n \mathbf{1} = \mathbf{1} + \ldots + \mathbf{1}$ be the disjoint union of n copies of $\mathbf{1}$. Since $+_n \mathbf{1}$ has no order relations, this means that every subset is an order ideal. So $J(+_n \mathbf{1}) = B_n$.*

Example 13 (n) *Note that every order ideal of $\mathbf{n}$ is of the form $\langle b \rangle$ for some $b \in \mathbf{n}$. Since $a \leq b$ iff $\langle a \rangle \subset \langle b \rangle$, this means that $J(\mathbf{n}) \simeq \mathbf{n+1}$.*

Example 14 (N) *Consider $\mathbf{N} = \{a, b, c, d\}$ defined at the beginning of lecture. We have*

$$J(\mathbf{N}) = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}, \{b, d\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b\}\}.$$

This gives the following Hasse diagram:

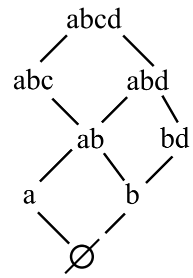


Figure 2: $J(\mathbf{N})$

Note that if $I_1, I_2 \subset P$ are order ideals, then $I_1 \cap I_2$ and $I_1 \cup I_2$ are order ideals. Since unions and intersections distribute over each other, this means that $J(P)$ is a distributive lattice for any poset P. The following result tells us that if a distributive lattice is finite, then it must be of this form.

Theorem 15 (Birkoff's Theorem) [1] *Let L be a finite distributive lattice. There exists a poset P unique up to isomorphism such that $L \simeq J(P)$.*

Proof:[Uniqueness] Given $J(P)$ we want to recover P. The subset of principal order ideals is a copy of P sitting inside $J(P)$. However, we want to describe this subset intrinsically without reference to P.

Fact 1. Let $S \subset J(P)$ be the subset of join irreducible elements. $S \simeq P$:

Let $T \subset J(P)$ be the subset of principal order ideals. Note that $T \simeq P$, because $\langle x \rangle \subset \langle y \rangle$ iff $x \leq_P y$. So it is enough to show that $S = T$.

($T \subset S$): Suppose that $\langle x \rangle = I_1 \cup I_2$ for $I_1, I_2 \in J(P)$. This means that either $x \in I_1$ or $x \in I_2$. So $\langle x \rangle \subset I_1$ or $\langle x \rangle \subset I_2$, implying $\langle x \rangle = I_1$ or $\langle x \rangle = I_2$.

($S \subset T$): Assume that $I \in J(P)$ is not principal. So we can find distinct maximal elements x and y in I. Note that $I - \{x\}$ and $I - \{y\}$ are order ideals. Since

$$I = (I - \{x\}) \cup (I - \{y\})$$

this implies that I is not join irreducible. □

We can now show uniqueness. Suppose that $J(P) \simeq L \simeq J(Q)$ for posets P and Q. This implies that the join irreducible elements of $J(P)$ and $J(Q)$ are isomorphic. So by Fact 1, we conclude that $P \simeq Q$. □

Proof:[Existence] From the proof of uniqueness, we have to show that $L = J(P)$ for P the join irreducible elements of L.

[1] This result is sometimes called the fundamental theorem of finite distributive lattices.

Fact 2. For $y \in L$ there exist $y_1, \ldots, y_n \in P$ such that $y = y_1 \vee \ldots \vee y_n$. For n minimal, this expression is unique up to permutation:

An element $y \in L$ is either join irreducible, or expressible as $y = y_1 \vee y_2$ for $y_1, y_2 < y$. Since L is finite, we see by induction that an element $y \in L$ can be written as $y = y_1 \vee \ldots \vee y_n$ for $y_i \in P$.

Choose n minimal with this property. Suppose that $y_1 \vee \ldots \vee y_n = y = z_1 \vee \ldots \vee z_n$ for $z_j, y_i \in P$. Note that $y \wedge z_i = z_i$ because $z_i \leq z_1 \vee \ldots \vee z_n = y$. By distributivity

$$z_i = z_i \wedge y = \bigvee_{j=1}^{n} z_i \wedge y_j.$$

Since $z_i \in P$, there exists some y_j such that $z_i = z_i \wedge y_j$. So $z_i \leq y_j$.

Suppose that $z_i, z_j \leq y_k$ for $i \neq j$. Since join is commutative and associative, we can assume w.l.o.g that $i = 1$ and $j = 2$. Replace $z_1 \vee z_2$ by y_k in $z_1 \vee \ldots \vee z_n$. So $y = y_k \vee (z_3 \vee \ldots \vee z_n)$ contradicting minimality of n. Therefore $\sigma \in S_n$.

Switching the roles of y_i and z_j, we obtain $\tau \in S_n$ such that $y_{\tau(i)} \leq z_{\sigma(i)} \leq y_i$. By minimality of n, we see that $y_{\tau(i)} = y_i$. This gives the result. □

Define a map $f : J(P) \ni I \mapsto \bigvee_{x \in I} x \in L$. By Fact 2, f is surjective.

Define a map $g : L \ni y \mapsto \bigcup_{y_i} \langle y_i \rangle \in J(P)$ where $y = y_1 \vee \ldots \vee y_n$ as in Fact 2. Since union commutes, this map is well-defined by Fact 2. An order ideal $I \subset P$ can be expressed as $I = \bigcup_i \langle y_i \rangle$ for $\{y_1, \ldots y_n\} \subset P$ the maximal elements of I. Note that

$$g(y_1 \vee \ldots \vee y_n) = \bigcup_i \langle y_i \rangle = I$$

because n is minimal by maximality of the y_i. Therefore gf is the identity on $J(P)$. This implies that f is injective.

(f order-preserving): Suppose that $I \subset I'$ for $I, I' \in J(P)$. We have

$$\begin{aligned} f(I') = \bigvee_{y \in I'} y &= (\bigvee_{y \in I} y) \vee (\bigvee_{y \in I' - I} y) \\ &= f(I) \vee z \end{aligned}$$

for some $z \in L$. This implies that $f(I') \geq f(I)$.

(g order-preserving): Suppose that for $I, I' \in J(P)$, we have $\bigvee_{x \in I} x \le \bigvee_{y \in I'} y$. We want to show that $I \subset I'$. Note

$$\begin{aligned}\bigvee_{x \in I} x \le \bigvee_{y \in I'} y \Longrightarrow x &\le \bigvee_{y \in I'} y \qquad \forall x \in I \\ \Longrightarrow x &= x \wedge x = (\bigvee_{y \in I'} y) \wedge x \\ \Longrightarrow x &= \bigvee_{y \in I'} (y \wedge x)\end{aligned}$$

by distributivity. Since x is join irreducible, this implies that $x = y \wedge x$ for some $y \in I'$. So $x \le y$ for some $y \in I'$. This means that $x \in I'$ because I' is an order ideal. Hence $I \subset I'$.

Therefore $J(P) \xrightarrow{\sim} L$. □

Note that we did not make full use of distributivity; we only needed to know that $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for $x, y, z \in L$. Hence property (i) of Definition 10 implies property (ii). This is a convenient fact for showing that a lattice is distributive, because only property (i) of Definition 10 needs to be checked.

We will end this lecture with a proposition that gives us another way of thinking about order ideals that will be useful when we try to compare posets.

Definition 16 *Given a poset P, define P^* to be the set P with reversed order relations i.e. $x \le_P y$ iff $x \ge_{P^*} y$.*

Proposition 17 $J(P) \simeq (\mathbf{2}^P)^*$.

Proof: Given a map $f \in \mathbf{2}^P$, let $\varphi(f) = f^{-1}(1) \subset P$. Since f is order-preserving, this implies that $\varphi(f) \in J(P)$. So we obtain a map $\varphi : \mathbf{2}^P \to J(P)$.

Suppose that $g \le_{\mathbf{2}^P} h$. If $h(x) = 1$ for some $x \in P$, then

$$g \le_{\mathbf{2}^P} h \Longrightarrow g(x) \le_P h(x) \Longrightarrow g(x) = 1.$$

So if $x \in \varphi(h)$ then $x \in \varphi(g)$, or equivalently $\varphi(h) \le_{J(P)} \varphi(g)$. Therefore φ is order-reversing.

Given an order ideal $I \subset P$, define a map $\psi(I) : P \to \mathbf{2}$ by

$$\psi(I)(x) := \begin{cases} 2 & \text{if } x \notin I \\ 1 & \text{if } x \in I \end{cases}.$$

Suppose that $x \le_P y$. There exist three cases:

$$x, y \in I \Longrightarrow \psi(I)(x) = 1 = \psi(I)(y),$$

$$x, y \notin I \Longrightarrow \psi(I)(x) = 2 = \psi(I)(y),$$
$$x \in I, y \notin I \Longrightarrow \psi(I)(x) = 1 <_{\mathbf{2}} 2 = \psi(I)(y).$$

So $\psi(I)(x)$ is order-preserving, and we obtain a map $\psi : J(P) \to \mathbf{2}^P$.

Suppose that $I \leq_{J(P)} I'$. For $x \in P$ there exist three cases:

$$x \in I, x \in I' \Longrightarrow \psi(I)(x) = 1 = \psi(I')(x),$$
$$x \notin I, x \in I' \Longrightarrow \psi(I)(x) = 2 \geq_{\mathbf{2}} 1 = \psi(I')(x),$$
$$x \notin I, x \notin I' \Longrightarrow \psi(I)(x) = 2 = \psi(I')(x).$$

So $\psi(I)(x) \geq_{\mathbf{2}} \psi(I')(x)$. Since x was arbitrary, this implies that $\psi(I) \geq_{\mathbf{2}^P} \psi(I')$. Therefore ψ is order-reversing.

We claim that φ and ψ are inverses. Consider $f \in \mathbf{2}^P$. By definition for $x \in P$,

$$\begin{aligned} [\psi\varphi(f)](x) &= [\psi(f^{-1}(1))](p) \\ &= \begin{cases} 2 & \text{if } x \notin f^{-1}(1) \\ 1 & \text{if } x \in f^{-1}(1) \end{cases} \\ &= \begin{cases} 2 & \text{if } f(x) = 2 \\ 1 & \text{if } f(x) = 1 \end{cases} = f(x) \end{aligned}$$

Therefore $\psi\varphi(f) = f$. Conversely, consider $I \in J(P)$. By definition

$$\varphi\psi(I) = \psi(I)^{-1}(1) = I.$$

Therefore $\varphi\psi(I) = I$. □