18.312: Algebraic Combinatorics — Lionel Levine

Problem Set 5

Due at the beginning of class on Thursday April 7, 2011

P20 Let P be the set of all faces of the hypercube $H_n = [0,1]^n \subset \mathbb{R}^n$, ordered by inclusion. (A face of H_n is a subset of the form $X_1 \times \cdots \times X_n$, where each X_i is either $\{0\}$, $\{1\}$ or $[0,1]$.)

(a) What is $\#P$?

(b) Show that P is graded with rank function $r(F) = \dim F$.

(c) How many elements of P have rank k?

(d) Show that $P = Q \times \cdots \times Q$ (with n factors) for some poset Q.

(e) Show that $\mathbf{1} \oplus P$ is a lattice. Is it distributive?

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P21 An inversion of a permutation $\pi \in S_n$ is a pair of indices $i < j$ such that $\pi(i) > \pi(j)$. Show that

$$[n]_q! = \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}$$

where $\mathrm{inv}(\pi)$ is the number of inversions of π.

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P22 Prove that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$ and that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

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P23 Consider the algebra $A = \mathbb{Q}[q]\langle x, y\rangle/(yx - qxy)$. That is, q commutes with everything in A, and x does not commute with y, but yx equals qxy in A. Prove the following identity in A:

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

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P24 The q-*exponential function* is defined as

$$\exp_q(x) = \sum_{n \geq 0} \frac{x^n}{[n]_q!}.$$

The q-derivative $D_q f$ of a function f is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}.$$

Show that $D_q \exp_q = \exp_q$.

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P25 Give a bijection between monic irreducible polynomials of degree n over $\mathbb{F}_q$, and primitive necklaces $(a_0, \ldots, a_{n-1})$ with beads $a_i \in [q]$. (Here we consider two necklaces equivalent if one is a rotation of the other.)

Hint for the last problem: you may find the following facts useful (and you may cite them without proof).

1. every irreducible polynomial $f(x) \in \mathbb{F}_q[x]$ of degree n has n roots in $\mathbb{F}_{q^n}$;
2. $\alpha \in \mathbb{F}_{q^n}$ is the root of an irreducible polynomial $f(x) \in \mathbb{F}_q[x]$ of degree n if and only if $\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{n-1}}$ are all distinct;
3. $\mathbb{F}_{q^n} - \{0\}$ is a cyclic group of order $q^n - 1$.