Problem Set 8

Due at the beginning of class on Tuesday May 3, 2011

P33 Let A be a symmetric $n \times n$ matrix with entries in $\{0, 1\}$, and let p be a prime number. Show that $\mathrm{Tr}(A^p - A)$ is divisible by p.

(Hint: Interpret A as an adjacency matrix.)

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P34 Find the number of closed paths of length ℓ in the 3-dimensional Hamming cube.

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P35 Suppose G is an undirected graph such that every vertex has the same degree d, and for all $\ell \geq 0$ the number of closed paths of length ℓ in G is

$$p_\ell = 5^\ell + 3 \cdot 5^{\ell/2} + 3 \cdot (-1)^\ell 5^{\ell/2} + 5 \cdot (-1)^\ell.$$

(a) How many vertices does G have?

(b) What is d?

(c) How many edges does G have?

(d) How many spanning trees does G have?

(e) (bonus) Does such a graph exist?

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P36 Let G be a directed graph on n vertices and let L be its Laplacian matrix. Let $R \subset [n]$. Find a combinatorial interpretation for the determinant of the matrix L_R obtained by crossing out the rows and columns of L indexed by R.

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P37 Let $G = (V, E)$ be a finite graph, and let $e = (v, w)$ be an edge of G. Show that the number of spanning trees $\kappa(G)$ obeys the recurrence

$$\kappa(G) = \kappa(G - e) + \kappa(G/e)$$

where $G - e = (V, E - \{e\})$ and G/e is the graph obtained from G by gluing together vertices v and w.