An algebraic analogue of a formula of Knuth

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Talk Outline

- Knuth's formula: generalizing n^{n-1}.
- ... with weights: generalizing $(x_1 + \ldots + x_n)^{n-1}$.
- ... with group structure: generalizing $(\mathbb{Z}/n\mathbb{Z})^{n-1}$.
- Recent developments!

Starting Point: Cayley's Theorem

- The number of rooted trees on n labeled vertices is n^{n-1}.
- Refinement: The number of trees with degree sequence $(d_1, \ldots, d_n)$ is the coefficient of $x_1^{d_1} \ldots x_n^{d_n}$ in

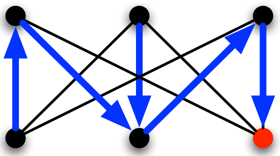
$$n x_1 \ldots x_n (x_1 + \ldots + x_n)^{n-2}.$$

- We can break this out by root:

$$\sum_{r=1}^{n} \prod_{i \neq r} x_i \quad \cdot \quad x_r (x_1 + \ldots + x_n)^{n-2}$$

outdegrees   indegrees

Oriented Spanning Trees



An oriented spanning tree of $K_{3,3}$.

- Let $G = (V, E)$ be a finite directed graph.
- An *oriented spanning tree* of G is a subgraph $T = (V, E')$ such that
  - one vertex, the root, has outdegree 0;
  - all other vertices have outdegree 1;
  - T has no oriented cycles $v_1 \to v_2 \to \cdots \to v_k \to v_1$.

Complexity of A Directed Graph

- The number

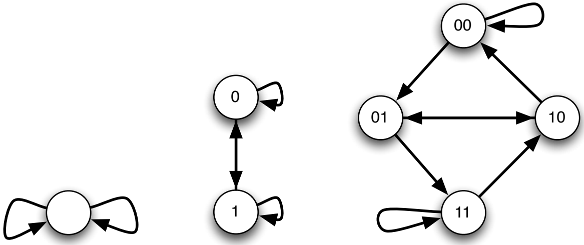
$$\kappa(G) = \# \text{ of oriented spanning trees of } G$$

is sometimes called the *complexity* of G.

- Examples:

$$\kappa(K_n) = n^{n-1}$$
$$\kappa(K_{m,n}) = (m+n)m^{n-1}n^{m-1}$$
$$\kappa(DB_n) = 2^{2^n-1}$$

The De Bruijn Graph DB_n



DB_0 DB_1 DB_2

- vertices $\{0,1\}^n$, edges $\{0,1\}^{n+1}$.
- The endpoints of the edge $e = b_1 \dots b_{n+1}$ are its prefix and suffix:

$$b_1 \dots b_n \xrightarrow{e} b_2 \dots b_{n+1}.$$

Directed Line Graphs

- $G = (V, E)$: finite directed graph
- $\mathtt{s}, \mathtt{t} : E \to V$
- Edge e is directed like this: $\mathtt{s}(e) \xrightarrow{e} \mathtt{t}(e)$
- The *directed line graph* $\mathcal{L}G = (E, E_2)$ of G has
  - Vertex set E, the edge set of G.
  - Edge set
$$E_2 = \{(e, f) \in E \times E \,|\, \mathtt{s}(f) = \mathtt{t}(e)\}.$$

$\bullet \xrightarrow{e} \bullet \xrightarrow{f} \bullet$

$(e, f) \in E_2$

$\bullet \xleftarrow{e} \bullet \xrightarrow{f} \bullet$

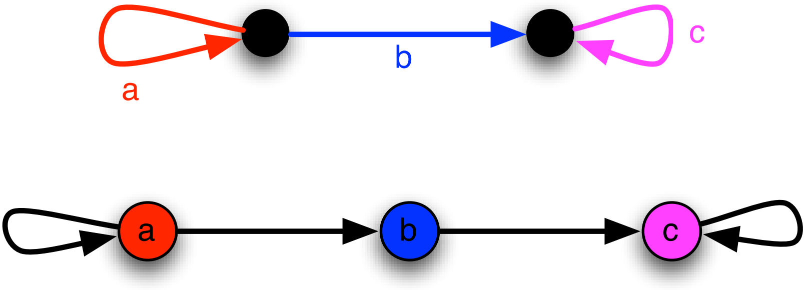
$(e, f) \notin E_2$

$\bullet \xrightarrow{e} \bullet$

$\bullet \xrightarrow{f} \bullet$

$(e, f) \notin E_2$

A Graph G and Its Directed Line Graph $\mathcal{L}G$



Examples of Directed Line Graphs

- $\vec{K}_n = \mathcal{L}$(one vertex with n loops).
- $\vec{K}_{m,n} =$
  $\mathcal{L}$(two vertices $\{a, b\}$ with m edges $a \to b$ and n edges $b \to a$).
- $DB_n = \mathcal{L}(DB_{n-1})$.
- Iterated line graphs: $\mathcal{L}^n G = (E_n, E_{n+1})$, where

$$E_n = \{\text{directed paths of } n \text{ edges in } G\}.$$

Spanning Tree Enumerators

- Let $(x_v)_{v \in V}$ and $(x_e)_{e \in E}$ be indeterminates, and let

$$\kappa^{edge}(G, \mathbf{x}) = \sum_T \prod_{e \in T} x_e$$

$$\kappa^{vertex}(G, \mathbf{x}) = \sum_T \prod_{e \in T} x_{\mathtt{t}(e)}$$

  The sums are over all oriented spanning trees T of G.
- Example:

$$\kappa^{vertex}(K_n, \mathbf{x}) = (x_1 + \cdots + x_n)^{n-1}.$$

Knuth's Formula

- $G = (V, E)$: finite directed graph with no sources
- outdegrees $a_1, \dots, a_n$
- indegrees $b_1, \dots, b_n \geq 1$
- $\mathcal{L}G$: the directed line graph of G
- **Theorem (Knuth, 1967).** For any edge e_* of G,

$$\kappa(\mathcal{L}G, e_*) = \alpha(G, e_*) \prod_{i=1}^{n} a_i^{b_i - 1}$$

where

$$\alpha(G, e_*) = \kappa(G, \mathtt{t}(e_*)) - \frac{1}{a_*} \sum_{\substack{\mathtt{t}(e) = \mathtt{t}(e_*) \\ e \neq e_*}} \kappa(G, \mathtt{s}(e)).$$

and a_* is the outdegree of $\mathtt{t}(e_*)$.

Weighted Knuth's Formula

- G : finite directed graph with no sources
- $\mathcal{L}G$: its directed line graph
- $b_1, \dots, b_n \geq 1$: the indegrees of G.
- **Theorem (L.)**

$$\kappa^{vertex}(\mathcal{L}G, \mathbf{x}) = \kappa^{edge}(G, \mathbf{x}) \prod_{i \in V} \left(\sum_{\mathrm{s}(e)=i} x_e \right)^{b_i - 1}.$$

- Both sides are polynomials in the edge variables x_e.

Specializing $x_e = 1$

- Complexity of a line graph:

$$\kappa(\mathcal{L}G) = \kappa(G) \prod_{i=1}^{n} a_i^{b_i - 1}.$$

- Examples:
  - G = one vertex with n loops, $\mathcal{L}G = K_n$, get n^{n-1}.
  - G = two vertices, $\mathcal{L}G = K_{m,n}$, get $(m+n)m^{n-1}n^{m-1}$.
  - $G = DB_{n-1}$, $\mathcal{L}G = DB_n$:

$$\begin{aligned} \kappa(DB_n) &= \kappa(DB_{n-1}) \cdot 2^{2^{n-1}} \\ &= \kappa(DB_{n-2}) \cdot 2^{2^{n-1}} \cdot 2^{2^{n-2}} \\ &= \dots \\ &= 2^{2^n - 1}. \end{aligned}$$

Rooted Version

- Fix an edge $e_* = (w_*, v_*)$ of G.
- Let b_* be the indegree of v_*.
- **Theorem** (L.) If $b_i \geq 1$ for all i, and $b_* \geq 2$, then

$$\kappa^{vertex}(\mathcal{L}G, e_*, \mathbf{x}) = x_{e_*} \kappa^{edge}(G, w_*, \mathbf{x}) \frac{\prod_{i \in V} \left(\sum_{\mathrm{s}(e)=i} x_e\right)^{b_i - 1}}{\sum_{\mathrm{s}(e)=v_*} x_e}.$$

Matrix-Tree Theorem

$$\kappa^{edge}(G,\mathbf{x}) = [t]\det(t \cdot Id - \Delta^{edge}).$$

$$\kappa^{vertex}(G,\mathbf{x}) = [t]\det(t \cdot Id - \Delta^{vertex}).$$

- Goal: relate Δ_G^{edge} with $\Delta_{\mathcal{L}G}^{vertex}$.

The Missing Link: Directed Incidence Matrices

- Consider the K-linear maps

$$A: \; K^E \to K^V, \qquad\qquad B : K^V \to K^E$$
$$e \mapsto \mathtt{t}(e) \qquad\qquad v \mapsto \sum_{\mathtt{s}(e)=v} x_e e.$$

Then

$$\Delta_G^{edge} = AB - D$$
$$\Delta_{\mathcal{L}G}^{vertex} = BA - D^{\mathcal{L}}$$

where D and $D^{\mathcal{L}}$ are the diagonal matrices

$$D(v) = \left(\sum_{\mathtt{s}(e)=v} x_e \right) v, \qquad D^{\mathcal{L}}(e) = \left(\sum_{\mathtt{s}(f)=\mathtt{t}(e)} x_f \right) e.$$

Intertwining Δ_G^{edge} and $\Delta_{\mathcal{L}G}^{vertex}$

$$A\Delta_{\mathcal{L}G}^{vertex} = A(BA - D^{\mathcal{L}}) = ABA - DA = (AB - D)A = \Delta_G^{edge} A$$

- In particular
$$\Delta_{\mathcal{L}G}^{vertex}(\ker A) \subset \ker A.$$
- Writing $K^E = \ker A \oplus \mathrm{Im}(A^T)$ puts $\Delta_{\mathcal{L}G}^{vertex}$ in block triangular form.

The Proof Falls Into Place

- Since G has no sources, $A: K^E \to K^V$ is onto.
  - So AA^T has full rank.
  - So $A: \mathrm{Im}(A^T) \to K^V$ is an isomorphism.
  - So $\det(\Delta^{vertex}_{\mathcal{L}G}|_{\mathrm{Im}(A^T)}) = \det \Delta^{edge}_{G} = \kappa(G, \mathbf{x})$.
- Eigenvalues of $\Delta^{vertex}_{\mathcal{L}G}|_{\ker A}$ are $\sum_{\mathtt{s}(e)=i} x_e$, each with multiplicity $b_i - 1$. □

Comparison with Knuth

- Knuth's formula involved the strange quantity

$$\alpha(G, e_*) = \kappa(G, \mathtt{t}(e_*)) - \frac{1}{a_*} \sum_{\substack{\mathtt{t}(e)=\mathtt{t}(e_*) \\ e \neq e_*}} \kappa(G, \mathtt{s}(e)).$$

- Why is it missing from our formulas?

The Unicycle Lemma

- A unicycle of G is an oriented spanning tree together with an outgoing edge from the root.
- By counting unicycles through v_* in two ways, we get:
- **Lemma.**

$$\kappa^{edge}(G, v_*, \mathbf{x}) \sum_{\mathrm{s}(e)=v_*} x_e = \sum_{\mathrm{t}(e)=v_*} \kappa^{edge}(G, \mathrm{s}(e), \mathbf{x})\, x_e.$$

- So Knuth's formula simplifies to

$$\kappa(\mathcal{L}G, e_*) = \frac{1}{a_*} \kappa(G, \mathrm{s}(e_*)) \prod_{i=1}^{n} a_i^{b_i - 1}.$$

The Sandpile Group of a Graph

- $K(G, v_*) \simeq \mathbb{Z}^{n-1}/\Delta\mathbb{Z}^{n-1}$, where

$$\Delta = D - A$$

  is the **reduced Laplacian** of G.

- Lorenzini '89/'91 ("group of components"), Dhar '90, Biggs '99 ("critical group"), Baker-Norine '07 ("Jacobian").
  - Directed graphs: Holroyd et al. '08
- **Matrix-tree theorem**:

$$\#K(G, v_*) = \det \Delta = \#\{\text{spanning trees of } G \text{ rooted at } v_*\}.$$

- Choice of sink: $K(G, v_*) \simeq K(G, v'_*)$ if G is Eulerian.

Maps Between Sandpile Groups

Theorem (L.) If G is Eulerian, then the map

$$\mathbb{Z}^E \to \mathbb{Z}^V$$
$$e \mapsto \mathtt{t}(e)$$

descends to a surjective group homomorphism

$$K(\mathcal{L}G, e_*) \to K(G, \mathtt{t}(e_*)).$$

Maps Between Sandpile Groups

Theorem (L.) If G is balanced k-regular, then the map

$$\mathbb{Z}^V \to \mathbb{Z}^E$$
$$v \mapsto \sum_{s(e)=v} e$$

descends to an isomorphism of groups

$$K(G) \simeq k\, K(\mathcal{L}G).$$

- Analogous to results of **Berget, Manion, Maxwell, Potechin and Reiner** on undirected line graphs. arXiv:0904.1246

The Sandpile Group of DB_n

- De Bruijn Graph $DB_n = \mathcal{L}^n$(a single vertex with 2 loops).
- **Theorem** (L.)

$$K(DB_n) = \bigoplus_{j=1}^{n-1} (\mathbb{Z}/2^j\mathbb{Z})^{2^{n-1-j}}.$$

- Generalized by **Bidkhori and Kishore** to k-ary De Bruijn graphs for any k.

Equating Exponents

- By counting spanning trees, we know that

$$\#K(DB_n) = \kappa(DB_n, v_*) = 2^{2^n - n - 1}.$$

- Now write

$$K(DB_n) = \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_4^{a_2} \oplus \mathbb{Z}_8^{a_3} \oplus \ldots \oplus \mathbb{Z}_{2^m}^{a_m}$$

for some nonnegative integers m and $a_1, \ldots, a_m$ satisfying

$$\sum_{j=1}^{m} j a_j = 2^n - n - 1. \qquad (1)$$

- By the previous theorem and inductive hypothesis

$$K(DB_{n-1}) \simeq 2K(DB_n)$$

$$\mathbb{Z}_2^{2^{n-3}} \oplus \mathbb{Z}_4^{2^{n-4}} \oplus \cdots \oplus \mathbb{Z}_{2^{n-2}} \simeq \mathbb{Z}_2^{a_2} \oplus \mathbb{Z}_4^{a_3} \oplus \ldots \oplus \mathbb{Z}_{2^{m-1}}^{a_m}.$$

- So $m = n - 1$ and $a_j = 2^{n-j-1}$. $\square$

A (Formerly) Open Problem From EC1

- In EC1, Stanley asks for a bijection

$$\{\text{pairs of binary De Bruijn sequences of order } n\}$$
$$\updownarrow$$
$$\{\text{all binary sequences of length } 2^n\}$$

- Both sets have cardinality 2^{2^n}.

Recent Developments

- In arXiv:0910.3442, **Bidkhori and Kishore** give a bijective proof of the weighted Knuth formula

$$\kappa^{vertex}(\mathcal{L}G, \mathbf{x}) = \kappa^{edge}(G, \mathbf{x}) \prod_{i \in V} \left(\sum_{\mathtt{s}(e)=i} x_e \right)^{b_i - 1}$$

  and use it to solve Stanley's problem!

- **Perkinson, Salter and Xu** give a surjective map

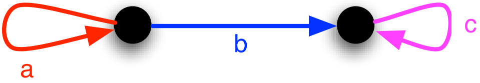
$$K(\mathcal{L}G, e_*) \to K(G, \mathtt{s}(e_*))$$

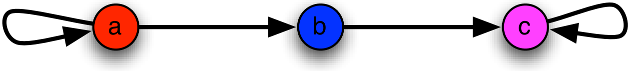
  even when G is not Eulerian.

Now What?

	unweighted	weighted
enumeration	$\kappa(G)$	$\kappa(G,\mathbf{x})$
algebra	$K(G)$	?

Thank You!





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- L., Sandpile groups and spanning trees of directed line graphs, *J. Comb. Theory A*, to appear. arXiv:0906.2809