Multi-Eulerian tours of directed graphs

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Abstract

Not every graph has an Eulerian tour. But every finite, strongly connected graph has a *multi-Eulerian tour*, which we define as a closed path that uses each directed edge at least once, and uses edges e and f the same number of times whenever $\mathrm{tail}(e) = \mathrm{tail}(f)$. This definition leads to a simple generalization of the BEST Theorem. We then show that the minimal length of a multi-Eulerian tour is bounded in terms of the Pham index, a measure of 'Eulerianness'.

Keywords: BEST theorem, coEulerian digraph, Eulerian digraph, Eulerian path, Laplacian, Markov chain tree theorem, matrix-tree theorem, oriented spanning tree, period vector, Pham index, rotor walk

In the following $G = (V, E)$ denotes a finite directed graph, with loops and multiple edges permitted. An **Eulerian tour** of G is a closed path that traverses each directed edge exactly once. Such a tour exists only if the indegree of each vertex equals its outdegree; the graphs with this property are called **Eulerian**. The BEST theorem (named for its discoverers: de Bruijn, Ehrenfest [4], Smith and Tutte [12]) counts the number of such tours. The purpose of this note is to generalize the notion of Eulerian tour and the BEST theorem to any finite, strongly connected graph G.

Definition 1. Fix a vector $\pi \in \mathbb{N}^V$ with all entries strictly positive. A **π-Eulerian tour** of G is a closed path that uses each directed edge e of G exactly $\pi_{\mathrm{tail}(e)}$ times.

Note that existence of a π-Eulerian tour implies that G is **strongly connected**: for each $v, w \in V$ there are directed paths from v to w and from w to v. We will show that,

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conversely, every strongly connected graph G has a π-Eulerian tour for suitable π. To do so, recall the BEST theorem counting $\mathbf{1}$-Eulerian tours of an Eulerian directed multigraph G. Write $\epsilon_\pi(G,e)$ for the number of π-Eulerian tours of G starting with a fixed edge e.

Theorem 1. (BEST [4, 12]) *A strongly connected multigraph G has a $\mathbf{1}$-Eulerian tour if and only if the indegree of each vertex equals its outdegree, in which case the number of such tours starting with a fixed edge e is*

$$\epsilon_{\mathbf{1}}(G,e) = \kappa_w \prod_{v\in V}(d_v-1)!$$

where d_v is the outdegree of v; vertex w is the tail of edge e, and κ_w is the number of spanning trees of G oriented toward w.

A **spanning tree oriented toward w** is a set of edges t such that w has outdegree 0 in t, each vertex $v \neq w$ has outdegree 1 in t, and t has no directed cycles. Let us remark that for a general directed graph the number κ_w of spanning trees oriented toward w depends on w, but for an Eulerian directed graph it does not (since $\epsilon_{\mathbf{1}}(G,e)$ does not depend on e).

The **graph Laplacian** is the $V \times V$ matrix

$$\Delta_{uv} = \begin{cases} d_v - d_{vv}, & u = v \\ -d_{vu} & u \neq v \end{cases}$$

where d_{vu} is the number of edges directed from v to u, and $d_v = \sum_u d_{vu}$ is the outdegree of v. By the matrix-tree theorem [10, 5.6.8], κ_w is the determinant of the submatrix of Δ omitting row and column w.

Thus, the BEST and matrix-tree theorems give a computationally efficient exact count of the $\mathbf{1}$-Eulerian tours of a directed multigraph. (In contrast, exact counting of *undirected* Eulerian tours is a #P-complete problem!)

Observing that the 'indegree=outdegree' condition in the BEST theorem is equivalent to $\Delta\mathbf{1} = \mathbf{0}$ where $\mathbf{1}$ is the all ones vector, we arrive at the statement of our main result.

Theorem 2. *Let $G = (V,E)$ be a strongly connected directed multigraph with Laplacian Δ, and let $\pi \in \mathbb{N}^V$. Then G has a π-Eulerian tour if and only if*

$$\Delta\pi = \mathbf{0}.$$

If $\Delta\pi = \mathbf{0}$, then the number of π-Eulerian tours starting with edge e is given by

$$\epsilon_\pi(G,e) = \kappa_w \prod_{v\in V} \frac{(d_v\pi_v - 1)!}{(\pi_v!)^{d_v-1}(\pi_v-1)!}$$

where d_v is the outdegree of v; vertex w is the tail of edge e, and κ_w is the number of spanning trees of G oriented toward w.

Note that the ratio on the right side is a multinomial coefficient and hence an integer.

The proof below is a straightforward application of the BEST theorem. The same proof device of constructing an Eulerian multigraph from a strongly connected graph was used in [2, Theorem 3.18] to relate the Riemann-Roch property of 'row chip-firing' to that of 'column chip-firing'. In the remainder of the paper we find the length of the shortest π-Eulerian tour (Theorem 5) and conclude with two mild generalizations: λ-Eulerian tours (Theorem 6) and π-Eulerian paths (Theorem 7).

Proof of Theorem 2. Define a multigraph $\widetilde{G}$ by replacing each edge e of G from u to v by π_u edges $e^1, \ldots, e^{\pi_u}$ from u to v. Since π has all positive entries, $\widetilde{G}$ is strongly connected. Each vertex v of $\widetilde{G}$ has outdegree $d_v \pi_v$ and indegree $\sum_{u \in V} \pi_u d_{uv}$, so $\widetilde{G}$ is Eulerian if and only if $\Delta\pi = \mathbf{0}$.

If $(e_1^{i_1}, \ldots, e_m^{i_m})$ is a $\mathbf{1}$-Eulerian tour of $\widetilde{G}$, then $(e_1, \ldots, e_m)$ is a π-Eulerian tour of G. Conversely, for each π-Eulerian tour of G, the occurrences of each edge f in the tour can be labeled with an arbitrary permutation of $\{1, \ldots, \pi_{\mathrm{tail}(f)}\}$ to obtain a $\mathbf{1}$-Eulerian tour of $\widetilde{G}$. Hence for a fixed edge e with $\mathrm{tail}(e) = w$,

$$\epsilon_\pi(G, e) \prod_{v \in V} (\pi_v!)^{d_v} = \epsilon_{\mathbf{1}}(\widetilde{G}, e^1) \pi_w.$$

The factor of π_w arises here from the label of the starting edge e, and the observation that $\epsilon_{\mathbf{1}}(\widetilde{G}, e^i)$ does not depend on i. In particular, G has a π-Eulerian tour if and only if $\widetilde{G}$ is Eulerian.

To complete the counting in the case when $\widetilde{G}$ is Eulerian, the BEST theorem gives the number of $\mathbf{1}$-Eulerian tours of $\widetilde{G}$ starting with e^1, namely

$$\epsilon_{\mathbf{1}}(\widetilde{G}, e^1) = \widetilde{\kappa}_w \prod_{v \in V} (d_v \pi_v - 1)!$$

where

$$\widetilde{\kappa}_w = \kappa_w \prod_{v \neq w} \pi_v \tag{1}$$

is the number of spanning trees of $\widetilde{G}$ oriented toward w, since each spanning tree of G oriented toward w gives rise to $\prod_{v \neq w} \pi_v$ spanning trees of $\widetilde{G}$.

We conclude that

$$\epsilon_\pi(G, e) = \pi_w \widetilde{\kappa}_w \prod_{v \in V} \frac{(d_v \pi_v - 1)!}{(\pi_v!)^{d_v}}$$

which together with (1) completes the proof. □

The watchful reader must now be wondering, is there a suitable vector π with *positive integer* entries in the kernel of the Laplacian? The answer is yes. Following Björner and Lovász, we say that a vector $\mathbf{p} \in \mathbb{N}^V$ is a **period vector** for G if $\mathbf{p} \neq \mathbf{0}$ and $\Delta\mathbf{p} = \mathbf{0}$. A period vector is **primitive** if the greatest common divisor of its entries is 1.

Lemma 3. *[5, Prop. 4.1] A strongly connected multigraph G has a unique primitive period vector π_G. All entries of π_G are strictly positive, and all period vectors of G are of the form $n\pi_G$ for $n = 1, 2, \dots$. Moreover, if G is Eulerian, then $\pi_G = \mathbf{1}$.*

Recall κ_v denotes the number of spanning trees of G oriented toward v. Broder [3] and Aldous [1] observed that $\kappa = (\kappa_v)_{v\in V}$ is a period vector! This result is sometimes called the 'Markov chain tree theorem'.

Lemma 4 ([1, 3])**.** $\Delta\kappa = \mathbf{0}$.

Lemmas 3 and 4 imply that the vector $\pi = \frac{1}{M_G}\kappa$ is the unique primitive period vector of G, where

$$M_G = \gcd\{\kappa_v \,:\, v \in V\}$$

is the greatest common divisor of the oriented spanning tree counts. Our next result expresses the minimal length of a multi-Eulerian tour in terms of M_G and the number

$$U_G = \sum_{v\in V} \kappa_v d_v$$

of **unicycles** in G (that is, pairs (t, e) where t is an oriented spanning tree and e is an outgoing edge from the root of t).

Theorem 5. *The minimal length of a multi-Eulerian tour in a strongly connected multigraph G is U_G/M_G.*

Proof. The length of a π-Eulerian tour is $\sum_{v\in V} \pi_v d_v$. By Theorem 2 along with Lemmas 3 and 4, there exists a π-Eulerian tour if and only if π is a positive integer multiple of the primitive period vector $\frac{1}{M_G}\kappa$. The result follows. □

A special class of multi-Eulerian tours are the simple rotor walks [9, 13, 7, 8, 11]. In a **simple rotor walk**, the successive exits from each vertex repeatedly cycle through a given cyclic permutation of the outgoing edges from that vertex. If G is Eulerian then a simple rotor walk on G eventually settles into an Eulerian tour which it traces repeatedly. More generally, if G is strongly connected then a simple rotor walk eventually settles into a π-Eulerian tour where π is the primitive period vector of G.

Trung Van Pham introduced the quantity M_G in [11] in order to count orbits of the rotor-router operation. In [6] we have called M_G the **Pham index** of G and studied the graphs with $M_G = 1$, which we called **coEulerian graphs**. The significance of M_G is not readily apparent from its definition, but we argue in [6] that M_G measures 'Eulerianness'. Theorem 5 makes this explicit, in that the minimal length of a multi-Eulerian tour depends inversely on M_G.

A consequence of Theorem 2 is that the number of π-Eulerian tours beginning with edge e does not depend on head(e). This can also be proved directly by cycling the tour to relate the number of tours starting with edge e to the total number of π-Eulerian tours:

$$\epsilon_\pi(G, e) = \frac{\pi_{\mathrm{tail}(e)} \sum_{f\in E} \epsilon_\pi(G, f)}{\sum_{v\in V} \pi_v d_v}.$$

We thank an anonymous referee for pointing out that the proof method of Theorem 2 also gives an efficient count of certain more general tours.

Definition 2. Fix a vector $\lambda \in \mathbb{N}^E$ with all entries strictly positive. A **$\boldsymbol{\lambda}$-Eulerian tour** is a closed path that uses each directed edge e exactly $\lambda(e)$ times.

Theorem 6. *Let $G = (V, E)$ be a strongly connected directed multigraph, and let $\lambda \in \mathbb{N}^E$. Then G has a λ-Eulerian tour if and only if*

$$\sum_{\mathrm{tail}(e)=v} \lambda_e = \sum_{\mathrm{head}(e)=v} \lambda_e \qquad \text{for all } v \in V. \tag{2}$$

If G has a λ-Eulerian tour, then the number of λ-Eulerian tours starting with a fixed edge e with tail w is

$$\det \widetilde{\Delta}_w \frac{\lambda_e \prod_{v \in V} (\widetilde{d}_v - 1)!}{\prod_{f \in E} (\lambda_f)!}$$

where $\widetilde{\Delta}_w$ is the submatrix omitting row and column w of the Laplacian of the multigraph $\widetilde{G}$ obtained by replacing each edge e of G from u to v by λ_e edges $e^1, \ldots, e^{\lambda_e}$ from u to v; and $\widetilde{d}_v = \sum_{\mathrm{tail}(e)=v} \lambda_e$ is the degree of v in $\widetilde{G}$.

Proof. If $(e_1^{i_1}, \ldots, e_\ell^{i_\ell})$ is a $\mathbf{1}$-Eulerian tour of $\widetilde{G}$, then $(e_1, \ldots, e_\ell)$ is a λ-Eulerian tour of G. Conversely, for each λ-Eulerian tour of G, the occurrences of each edge f in the tour can be labeled with an arbitrary permutation of $\{1, \ldots, \lambda_f\}$ to obtain a $\mathbf{1}$-Eulerian tour of $\widetilde{G}$. Hence for a fixed edge e with $\mathrm{tail}(e) = w$,

$$\epsilon_\lambda(G, e) \prod_{f \in E} (\lambda_f)! = \epsilon_{\mathbf{1}}(\widetilde{G}, e^1) \lambda_e.$$

In particular, G has a λ-Eulerian tour if and only if $\widetilde{G}$ is Eulerian, which happens if and only if (2) holds.

To complete the counting in the case when $\widetilde{G}$ is Eulerian, the BEST theorem gives the number of $\mathbf{1}$-Eulerian tours of $\widetilde{G}$ starting with e^1, namely

$$\epsilon_{\mathbf{1}}(\widetilde{G}, e^1) = \det \widetilde{\Delta}_w \prod_{v \in V} (\widetilde{d}_v - 1)!$$

where $\det \widetilde{\Delta}_w$ is the number of spanning trees of $\widetilde{G}$ oriented toward w by the matrix-tree theorem. $\square$

So far we have assumed that G is strongly connected. For our last result we drop this assumption, and count π-Eulerian paths which are permitted to start and end at different vertices.

Definition 3. Fix $\pi \in \mathbb{N}^V$ with all entries strictly positive, and vertices $a, z \in V$. A **$\boldsymbol{\pi}$-Eulerian path** from a to z is a path $a = e_1, \ldots, e_m = z$ in which each edge e appears exactly $\pi_{\mathrm{tail}(e)}$ times.

Theorem 7. *Let $G = (V, E)$ be a directed multigraph with Laplacian Δ, let $\pi \in \mathbb{N}^V$ and fix vertices $a \neq z$. Then G has a π-Eulerian path from a to z if and only if $(V, E \cup (z, a))$ is strongly connected and*

$$\Delta\pi = 1_a - 1_z.$$

If G has a π-Eulerian path from a to z, then the number of such paths is

$$\epsilon_\pi(G, a \to z) = \kappa_z \frac{(d_z \pi_z)!}{(\pi_z)!^{d_z}} \prod_{v \in V - \{z\}} \frac{(d_v \pi_v - 1)!}{(\pi_v!)^{d_v - 1}(\pi_v - 1)!}. \tag{3}$$

Proof. Let G' be the multigraph obtained from G by adding a new vertex w with one edge (z, w), one edge (w, a) and $\pi_z - 1$ edges (w, z). Set $\pi_w = 1$. Given a π-Eulerian tour of G', omitting all edges incident to w yields a π-Eulerian path from a to z in G. Conversely, any π-Eulerian path from a to z in G can be augmented to a π-Eulerian tour of G' beginning with the edge (w, a) (and necessarily ending with edge (z, w)) by inserting $\pi_z - 1$ detours from z to w and back. (Here we have used $a \neq z$; in the case $a = z$ we would need to set $\pi_w = \pi_z$.) This insertion can be performed in $\binom{d_z\pi_z + \pi_z - 1}{\pi_z - 1}(\pi_z - 1)!$ possible ways. Hence

$$\epsilon_\pi(G', (w, a)) = \epsilon_\pi(G, a \to z) \binom{d_z\pi_z + \pi_z - 1}{\pi_z - 1} (\pi_z - 1)!.$$

In particular, G has a π-Eulerian path from a to z if and only if G' has a π-Eulerian tour. By Theorem 2, this happens if and only if G' is strongly connected and $\Delta'\pi = \mathbf{0}$, where Δ' is the Laplacian of G'; equivalently, $(V, E \cup (z, a))$ is strongly connected and $\Delta\pi = 1_a - 1_z$.

For the count, since the spanning trees of G' oriented toward w are in bijection with the spanning trees of G oriented toward z, we obtain from Theorem 2

$$\epsilon_\pi(G', (w, a)) = \kappa_z \prod_{v \in V \cup \{w\}} \frac{(d'_v \pi_v - 1)!}{(\pi_v!)^{d'_v - 1}(\pi_v - 1)!}$$

where d'_v is the outdegree of v in G'. For $v \notin \{w, z\}$ we have $d'_v = d_v$. Since $d'_w = \pi_z$ and $\pi_w = 1$, the ratio on the right side is just $(\pi_z - 1)!$ when $v = w$. Since $d'_z = d_z + 1$, we end up with

$$\epsilon_\pi(G, a \to z) = \kappa_z \binom{d_z\pi_z + \pi_z - 1}{\pi_z - 1}^{-1} \frac{(d_z\pi_z + \pi_z - 1)!}{(\pi_z!)^{d_z}(\pi_z - 1)!} \prod_{v \in V - \{z\}} \frac{(d_v \pi_v - 1)!}{(\pi_v!)^{d_v - 1}(\pi_v - 1)!}$$

which simplifies to (3). □

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