WHAT IS a sandpile?

Lionel Levine and James Propp

An abelian sandpile is a collection of indistinguishable chips distributed among the vertices of a graph. More precisely, it is a function from the vertices to the nonnegative integers, indicating how many chips are at each vertex. A vertex is called *unstable* if it has at least as many chips as its degree, and an unstable vertex can *topple* by sending one chip to each neighboring vertex. Note that toppling one vertex may cause neighboring vertices to become unstable. If the graph is connected and infinite, and the number of chips is finite, then all vertices become stable after finitely many topplings. An easy lemma says that the final stable configuration is independent of the order of topplings (this is the reason for calling sandpiles "abelian"). For instance, start with a large pile of chips at the origin of the square grid $\mathbb{Z}^2$ and perform topplings until every vertex is stable. The process gives rise to a beautiful large-scale pattern (Figure 1). More generally, one obtains different patterns by starting with a constant number $h \leq 2d-2$ of chips at each site in $\mathbb{Z}^d$ and adding n chips at the origin; see Figure 3 for two examples.

Sandpile dynamics have been invented numerous times, attached to such names as chip-firing, the probabilistic abacus, and the dollar game. The name "sandpile" comes from statistical physics, where the model was proposed in a famous 1987 paper of Bak, Tang and Wiesenfeld as an example of *self-organized criticality*, or the tendency of physical systems to drive themselves toward critical, barely stable states. In the original BTW model, chips are added at random vertices of an $N \times N$ box in $\mathbb{Z}^2$. Each time a chip is added, it may cause an avalanche of topplings. If this avalanche reaches the boundary, then topplings at the boundary cause chips to disappear from the system. In the stationary state, the distribution of avalanche sizes has a power-law tail: very large avalanches occur quite frequently (e.g., the expected number of topplings in an avalanche goes to infinity with N).

To any finite connected graph G we can associate an abelian group $K(G)$, called the *sandpile group*. This group is an isomorphism invariant of the graph and reflects certain combinatorial information about the graph. To define the group, we single out one vertex of G as the *sink* and ignore chips that fall into the sink. The operation of addition followed by stabilization gives the set M of all stable sandpiles on G the structure of a commutative monoid. An *ideal* of M is a subset $J \subset M$ satisfying $\sigma J \subset J$ for all $\sigma \in M$. The sandpile group $K(G)$ is the minimal ideal of M (i.e., the intersection of all ideals). The minimal ideal of a finite commutative monoid is always a group. (We encourage readers unfamiliar with this remarkable fact to prove it for themselves.)

One interesting feature of constructing a group in this manner is that it is not at all obvious what the identity element is! Indeed, for many graphs G the identity element of $K(G)$ is a highly nontrivial object with intricate structure (Figure 2).

To realize the sandpile group in a more concrete way, we can view sandpiles σ as elements of the free abelian group $\mathbb{Z}^V$, where V is the set of non-sink vertices of G. Toppling a vertex v corresponds to adding the vector Δ_v to σ, where

$$\Delta_{v,w} = \begin{cases} -d(v) & \text{if } v = w, \\ 1 & \text{if } v \sim w, \\ 0 & \text{otherwise.} \end{cases}$$

Here $v \sim w$ denotes adjacency in G, and $d(v)$ is the degree of vertex v. This observation suggests that we view two vectors $\sigma, \tau \in \mathbb{Z}^V$ as equivalent if and only if their difference lies in the $\mathbb{Z}$-linear span of the vectors Δ_v.

The sandpiles lying in the minimal ideal of M are called *recurrent*. It turns out that each equivalence class in $\mathbb{Z}^V$ contains exactly one recurrent sandpile, and hence

$$K(G) = \mathbb{Z}^V / \Delta\, \mathbb{Z}^V \; .$$

The matrix $\Delta = (\Delta_{v,w})$ is called the *reduced Laplacian* of G (it is reduced because it does not include the row and column corresponding to the sink vertex). According to the *matrix-tree theorem*, the determinant $\det \Delta$ counts the number of spanning trees of G. This determinant is also the index of the subgroup $\Delta\mathbb{Z}^V$ in $\mathbb{Z}^V$, and so the order of the sandpile group equals the number of spanning trees.

A refinement relates sandpiles to the Tutte polynomial $T(x, y)$ of G. The number of spanning trees of G equals $T(1, 1)$. By a theorem of Merino López, $T(1, y)$ equals the sum of $y^{|\sigma|-m+\delta}$ over all recurrent sandpiles σ, where δ is the degree of the distinguished sink vertex, m is the number of edges of G, and $|\sigma|$ denotes the number of chips in σ.

The sandpile group gives algebraic manifestations to many classical enumerations of spanning trees. For example, Cayley's formula n^{n-2} for the number of spanning trees of the complete graph K_n becomes

$$K(K_n) = (\mathbb{Z}_n)^{n-2}.$$

The formula $m^{n-1}n^{m-1}$ for the number of spanning trees of the complete bipartite graph becomes

$$K(K_{m,n}) = \mathbb{Z}_{mn} \times (\mathbb{Z}_m)^{n-2} \times (\mathbb{Z}_n)^{m-2}.$$

The name "sandpile group" is due to Dhar, who used the group to analyze the BTW sandpile model.

A deep analogy between graphs and algebraic curves can be traced back implicitly to a 1970 theorem of Raynaud, which relates the component group of the Neron model of the Jacobian of a curve to the Laplacian matrix of an associated graph. In this analogy, the sandpile group of the graph plays a role analogous to the Picard group of the curve. Many of the authors who explored this analogy chose different names for the sandpile group, including "group of components" (Lorenzini), "Jacobian group" (Bacher et al.) and "critical group" (Biggs). Recent work of Baker and Norine carries the analogy further by proving a Riemann-Roch theorem for graphs.

The *odometer* of a sandpile σ is the function on vertices defined by

$$\begin{aligned} u(v) = \#\text{ of times } v \text{ topples} \\ \text{during the stabilization of } \sigma. \end{aligned}$$

The final stable configuration τ is given in terms of σ and u by

$$\tau = \sigma + \Delta u.$$

In particular, u obeys the inequalities

$$u \geq 0, \tag{1}$$

$$\sigma + \Delta u \leq d - 1. \tag{2}$$

One can show that the sandpile toppling rule implies a kind of *least action principle*: the odometer function is the pointwise minimum of all integer-valued functions u satisfying (1) and (2).

The least action principle says that sandpiles are "lazy" in a rather strong sense: even if we allow "illegal" toppling sequences that result in some vertices having a negative number of chips, we cannot stabilize σ in fewer topplings than occur in the sandpile dynamics. What is more, sandpiles are locally lazy: not only is the total number of topplings minimized, but each vertex does the minimum amount of work required of it to produce a stable final configuration.

The least action principle characterizes the odometer function as the solution to a type of variational problem in partial differential equations called an *obstacle problem*. The problem takes its name from an equivalent formulation in which one is given a function called the *obstacle* and asked to find the smallest superharmonic function lying above it.

The obstacle problem for the sandpile odometer has one extra wrinkle, which is the constraint that u be integer valued. Relaxing this constraint yields the odometer function for a different model called the *divisible sandpile*, in which the discrete chips are replaced by a continuous amount of mass which may be subdivided arbitrarily finely during topplings. The divisible sandpile has dramatically different behavior: starting with mass m at the origin in $\mathbb{Z}^2$, one obtains a region A_m of fully occupied sites, bordered by a strip of partly filled sites. The set A_m is very nearly circular, reflecting the rotational symmetry of the continuous Laplacian. Amazingly, the anisotropy as well as the intricate patterns of Figure 1 arise entirely from the extra integrality constraint.

Two fundamental features of sandpiles in lattices $\mathbb{Z}^d$ remain unexplained by theorems. One is *scale invariance*: large sandpiles look like scaled up small sandpiles. The picture in Figure 1, rescaled by a factor of $1/\sqrt{n}$, appears to have a limit as $n \to \infty$. The limit is a function f on the unit square $[0, 1]^2$ which is locally constant on an open dense subset. Each region where f is constant corresponds to a patch on which the sandpile configuration is periodic. The second unexplained feature is *dimensional reduction*: d-dimensional slices of $(d+1)$-dimensional sandpiles look like d-dimensional sandpiles, except in a region near the origin. Figure 3 compares a sandpile in $\mathbb{Z}^2$ with a 2-dimensional slice of a sandpile in $\mathbb{Z}^3$.

As a way of measuring avalanches, Dhar considered the odometer function associated with the operation of adding a single chip to a sandpile. Starting from the stationary state and adding a single chip at v, let $u_v(w)$ be the expected number of times w topples. When the system stabilizes, it is again in the stationary state, so the expected net change in height from topplings is $\Delta u_v(w) = -\delta_{v,w}$ (here δ is Kronecker's delta). In other words,

$$u_v(w) = (-\Delta^{-1})_{v,w}.$$

The entry $(-\Delta^{-1})_{v,w}$ of the inverse reduced Laplacian matrix has a natural interpretation in terms of random walks: it is the expected number of visits to w by a random walk on G started at v and stopped when it first visits the sink. For example, if G is the cube of side length n in $\mathbb{Z}^d$ $(d \geq 3)$ with sink at the boundary of the cube, then this expectation has order $|v-w|^{2-d}$ for v, w away from the boundary. Summing over w, we see that the expected number of topplings diverges as $n \to \infty$. The situation is even more extreme for $d = 2$: the expected number of times each individual site near v topples goes to infinity with n.

References

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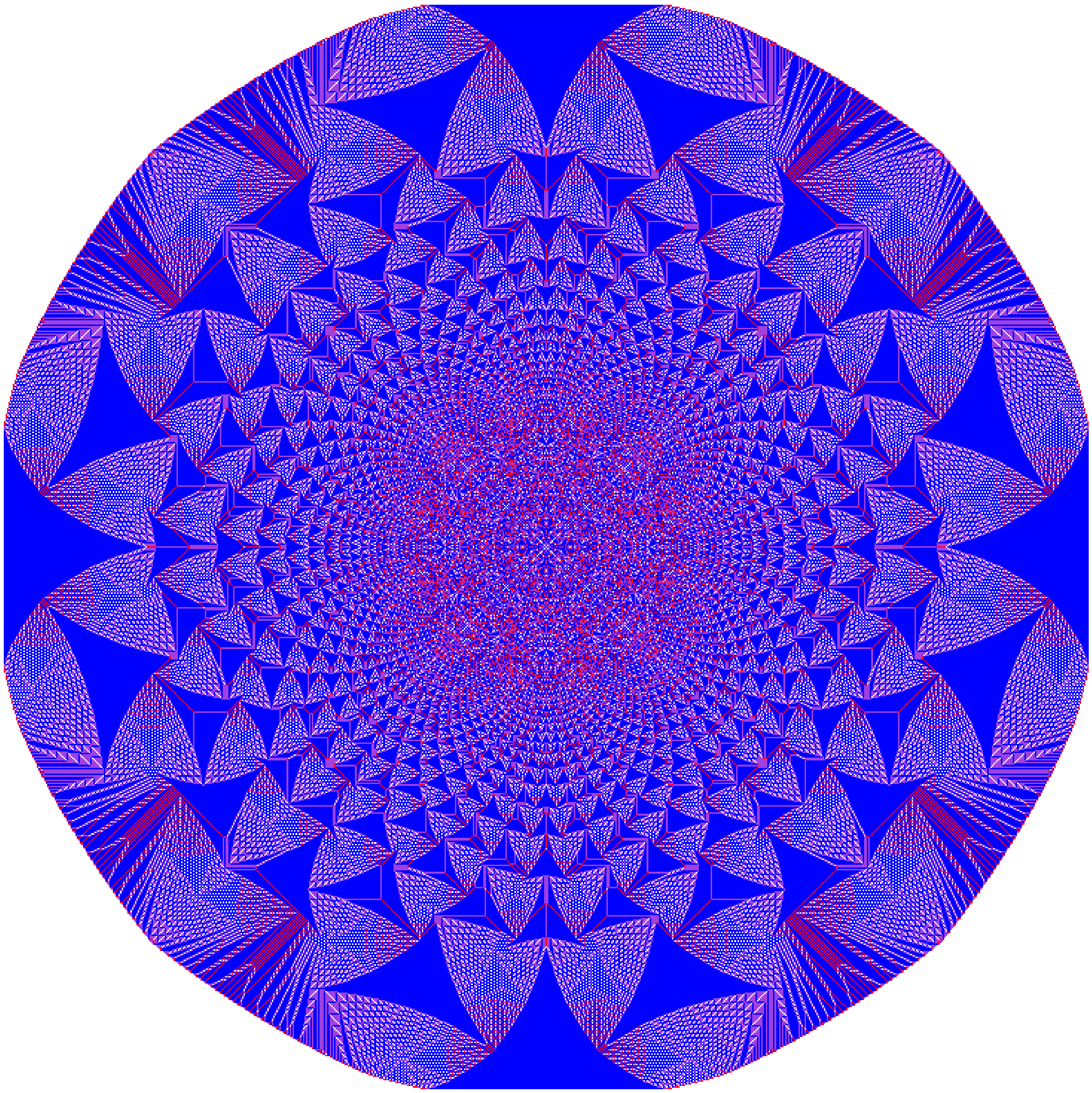

FIGURE 1. Stable sandpile of $n = 10^6$ chips in $\mathbb{Z}^2$. Color scheme: sites colored blue have 3 chips, purple 2 chips, red 1 chip, white 0 chips.

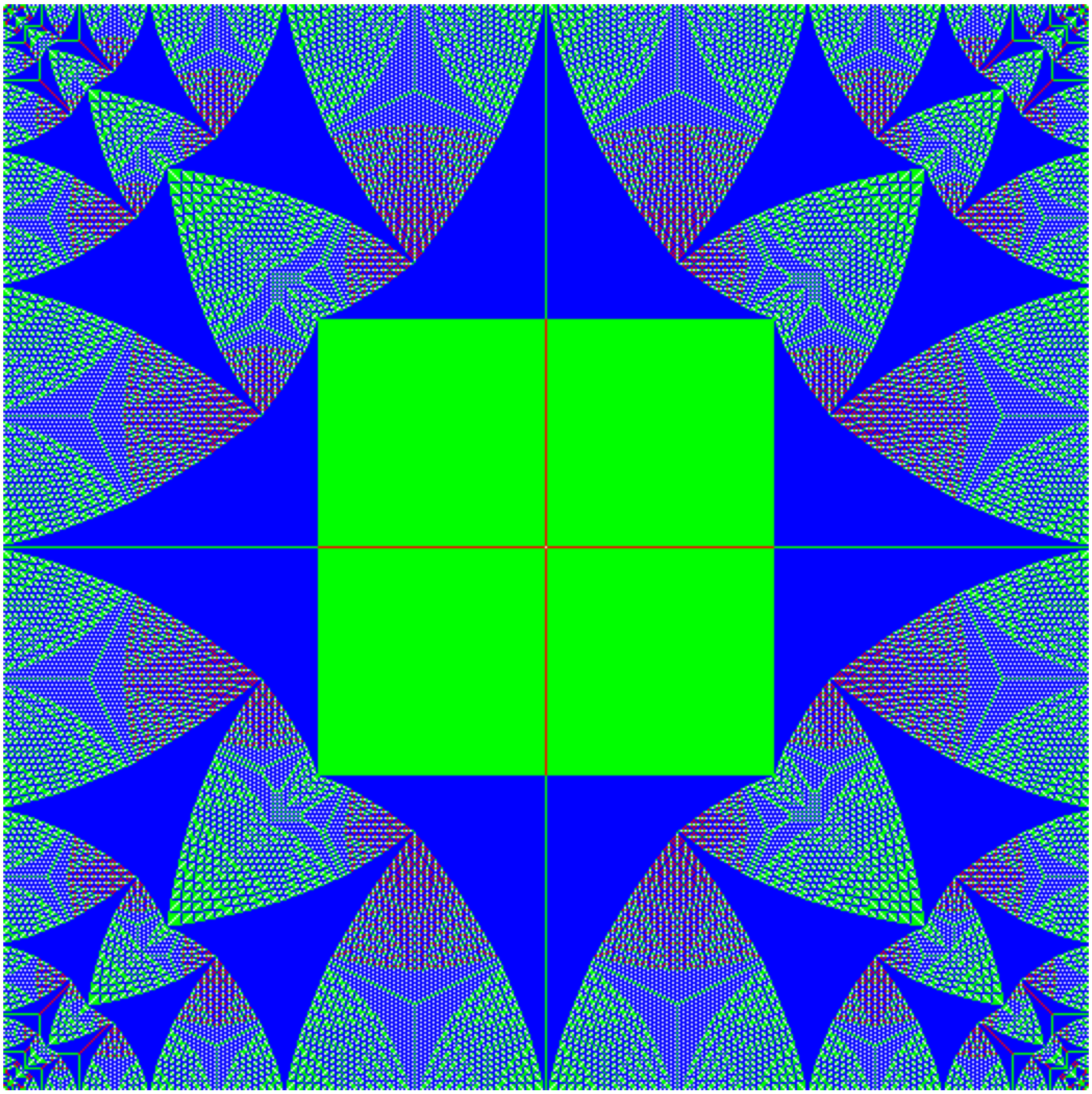

FIGURE 2. Identity element of the sandpile group of the 521×521 square grid graph, with sink at the boundary. Color scheme: sites colored blue have 3 chips, green 2 chips, red 1 chip, orange 0 chips.

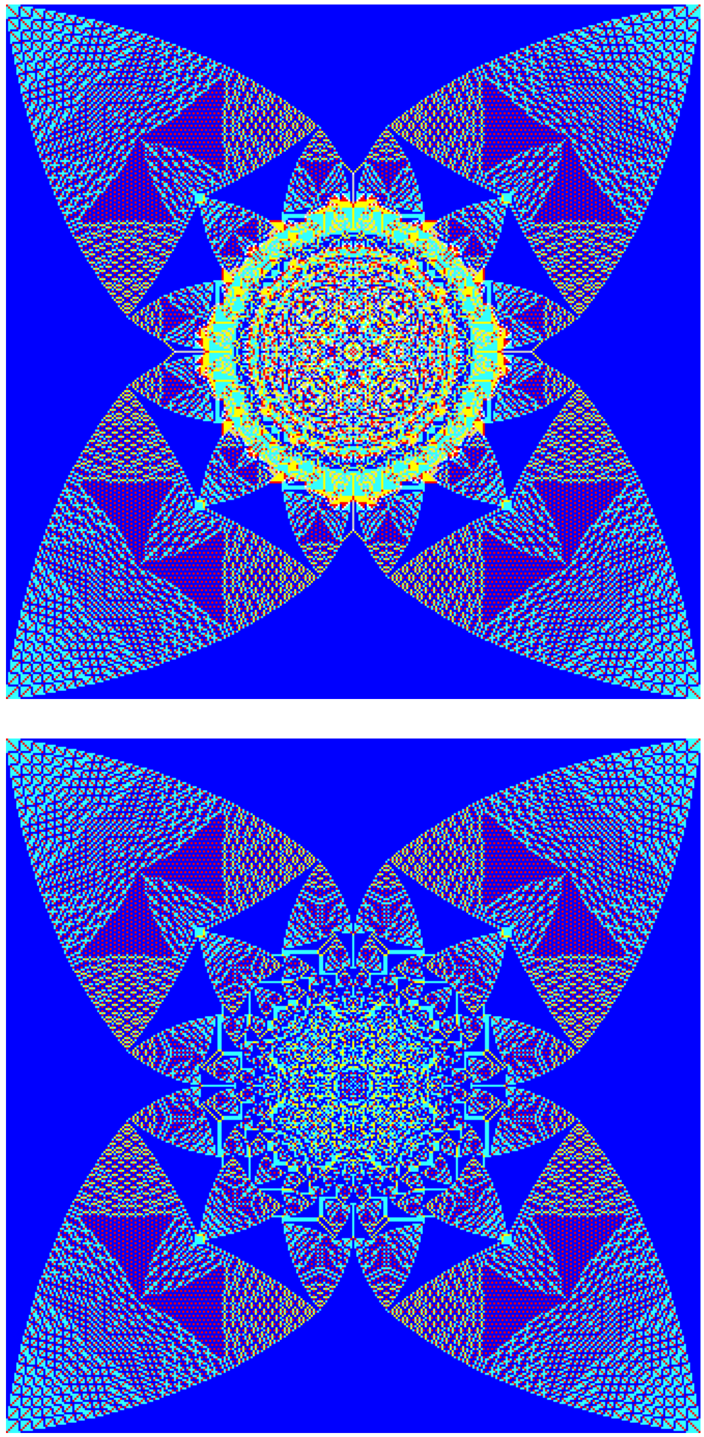

FIGURE 3. Top: A two-dimensional slice through the origin of the sandpile of $n = 5 \cdot 10^6$ particles in $\mathbb{Z}^3$ on background height $h = 4$. Bottom: The sandpile of $m = 47465$ particles in $\mathbb{Z}^2$ on background height $h = 2$. Color scheme on left: sites colored blue have 5 particles, turquoise 4, yellow 3, red 2, gray 1, white 0. On right: blue 3 particles, turquoise 2, yellow 1, red 0.